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Applications of Calculus of Variations
and Optimal Control to Space
Rendezvous

Ἔν οἶδα ὅτι δέν οἶδα
(The only thing I know is that I don't know anything)
*Socrates*¹

We have all heard the old adage, takeoffs are optional, landings are
mandatory.
We rephrase it as takeoffs are optional, meetings are mandatory.

¹ Socrates of Athens (*Σωκράτης ο Αθηναίος*) ~470–~399 (BC).

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Preface

This book is under construction. Be patient, please!

Throughout this book we use the following notations: $\mathbb{N} = \{0, 1, 2, \dots\}$ for the set of natural numbers, $\mathbb{N}^* = \{1, 2, \dots\}$ for the set of nonzero natural numbers, $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$ for the set of integer numbers, $\mathbb{Q} = \{p/q \mid p, q \in \mathbb{Z}, q \neq 0\}$ for the set of rational numbers, \mathbb{R} for set of real numbers, \mathbb{R}^m for the real m -dimensional Euclidean² space, $m \in \mathbb{N}$. We note that $\mathbb{R}^0 = \{0\}$ and $\mathbb{R}^1 = \mathbb{R}$. $\mathbb{R}_+ = [0, +\infty[$ is the set of nonnegative real numbers, $\overline{\mathbb{R}} = [-\infty, +\infty]$ is the set of extended real numbers, and $\overline{\mathbb{R}}_+ = [0, +\infty]$ is the set of extended nonnegative real numbers.

All the figures and calculations are done by *Mathematica*[®].

Most of the stuff of the present booklet are taken from [126].

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² Euclid of Alexandria, $\sim 325 - \sim 265$ (BC)